

# GravitoMagnetic Field in Tensor-Vector-Scalar Theory

Qasem Exirifard

*School of Physics, Institute for Research in Fundamental Sciences, Tehran 19538-33511, Iran\**

We consider the perturbative regime of the TeVeS theory. We then compute the gravitomagnetic field that a slow moving mass distribution produces in its weak Newtonian regime. We report that the consistency of the TeVeS gravitomagnetic field near the Earth space-time geometry with the Lunar Laser Ranging measurements requires the scalar coupling  $\frac{k}{4\pi}$  to be weaker than  $3.3 \times 10^{-4}$  at the standard confidence level. This bound beside the astronomical data demands that the dimension full parameter of the theory,  $l$ , holds  $l > (1.65 \pm 0.35) \times 10^{25} m$ .

PACS numbers: 04.20.-q, 04.50.Kd, 95.35.+d

The missing mass problem in the galaxies can be solved either by assuming the presence of dark matter, a form of matter yet not included and seen in the standard model of the elementary particles, or altering the newtonian dynamics and gravity. The  $\Lambda$ CDM model buys the first approach. The success of the  $\Lambda$ CDM theory has stemmed various searches to directly/indirectly observe dark matter particles and their signatures which can not be produced by altering newtonian dynamics/gravity. However no conclusive signals for the dark matter particles yet have been reported.

The second approach started by the Milgrom theory of the modified newtonian dynamics (MOND)[1], then changed to the aquadratic Lagrangian model for gravity [2] (AQUAL). This approach also has signatures that can not be produced by the dark matter paradigm, including the detection of the gravitomagnetic force in the MOND regime [4]. Now a generally covariant realization of the AQUAL model, the TeVeS theory [3], is known to represents the modified gravity's approach.

TeVeS equations of motion are more evolved compared to those of the Einstein-Hilbert theory. Though its spherically static solution [5] - analog to the Schwarzschild solution- is known, the exact Kerr-like solution of the theory so far has not been reported. In one hand no solution of TeVeS possessing non-trivial gravitomagnetic field is known. In the other hand, we have acquired very precise empirical knowledge on the gravitomagnetic field around the space-time geometry of the earth. These include the Lunar Laser measurements [6] and the gravity probe B [7]. In this paper, we investigate how measuring the gravitomagnetic force near the earth constraints the TeVeS parameters. In so doing, we first review the field equations of the TeVeS theory. We next solve these equations at the linear order around the space-time geometry of the earth. We show that the TeVeS gravitomagnetic force is less than the Einstein-Hilbert gravitomagnetic field by the factor of  $\frac{3k}{4\pi}$ . Noting that the LLR measurements require the consistency with the Einstein gravitomagnetic field with the accuracy of 0.1% we bound the scalar coupling  $\frac{k}{4\pi}$  to be weaker than  $2.3 \times 10^{-4}$  at the standard confidence level. This value of  $k$  constraints the dimension-full parameter of the TeVeS to  $l > (1.65 \pm 0.35) \times 10^{25} m$ . These are the first constraints on  $l$  and  $k$  parameters of the theory.

## I. EQUATIONS OF MOTION OF TEVES

The dynamical fields in the Tensor Vector Scalar (TeVeS) theory [3] are the Einstein metric  $g_{ab}$ , a time-like vector  $A_a$ , and a scalar  $\phi$ . There is also a non-dynamical scalar field  $\sigma$ . The Einstein metric couples to the physical energy momentum tensor represented by  $T_{ab}$ . The physical metric is defined by

$$\tilde{g}_{ab} = e^{-2\phi} g_{ab} - 2A_a A_b \sinh(2\phi) \quad (1)$$

The vector field is forced to be the unite timelike vector with respect to the Einstein metric

$$\tilde{g}^{ab} A_a A_b = 1 \quad (2)$$

We assume that matter is an ideal fluid. So the physical energy momentum tensor takes the form of

$$T_{ab} = \rho u_a u_b + p(\tilde{g}_{ab} + u_a u_b) \quad (3)$$

where  $\rho$  is the proper energy,  $p$  is the pressure and  $u_a$  is the 4-velocity, all three expressed in the physical metric.

---

\*Electronic address: exir@theory.ipm.ac.ir

The metric equation of motion reads

$$G_{ab} = 8\pi G(T_{ab} + (1 - e^{-4\phi})A^\mu T_{\mu(a}A_{b)} + \tau_{ab}) + \Theta_{ab} \quad (4)$$

where  $G_{ab}$  is the Einstein tensor constructed out from the metric and its derivatives, round bracket represent symmetrization with respect to indices, e.g.  $A_{(a}B_{b)} = A_a B_b + A_b B_a$  while  $\tau_{ab}$  and  $\Theta_{ab}$  stand for the contribution of the scalar and vector fields:

$$\tau_{ab} \equiv \sigma^2 \left( \phi_{,a} \phi_{,b} - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} g_{ab} - A^\mu \phi_{,\mu} (A_{(a} \phi_{,b)} - \frac{1}{2} A^\nu \phi_{,\nu} g_{ab}) \right) - \frac{1}{4} G l^{-2} \sigma^4 F(kG\sigma^2) g_{ab}, \quad (5)$$

$$\Theta_{ab} \equiv K \left( g^{\mu\nu} F_{\mu a} F_{\nu b} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} g_{ab} \right) - \lambda A_a A_b \quad (6)$$

where  $F_{ab}$  represent the field strength of the vector field

$$F_{ab} = \partial_a A_b - \partial_b A_a \quad (7)$$

and  $\lambda$  is a Lagrange multiplier,  $K$  and  $k$  are respectively the vector and scalar couplings,  $l$  is a parameter and  $F$  stands for the TeVeS function. It is useful to define a function  $\mu(y)$  by

$$-\mu F(\mu) - \frac{1}{2} \mu^2 F'(\mu) = y \quad (8)$$

using which expresses the equations of the scalar fields to

$$kG\sigma^2 = \mu(kl^2 h^{\mu\nu} \phi_{,\mu} \phi_{,\nu}) \quad (9)$$

$$\nabla_\beta (\mu(kl^2 h^{\mu\nu} \phi_{,\mu} \phi_{,\nu}) h^{\alpha\beta} \phi_{,\alpha}) = kG[g^{\mu\nu} + (1 + e^{-4\phi})A^\mu A^\nu] T_{\mu\nu} \quad (10)$$

where

$$h^{ab} \equiv g^{ab} - A^a A^b \quad (11)$$

The equation of motion for the vector field reads

$$K \nabla_b F^{ab} + \lambda A^a + 8\pi G \sigma^2 A^b \nabla_b \phi \nabla^a \phi = 8\pi G (1 - e^{-4\phi}) A_b T^{ab} \quad (12)$$

Recalling (2), one can derive the equation of motion for the Lagrange multiplier by contracting the vector equation with  $A_a$ . The  $\lambda$ 's equation is a linear algebraic equation in term of  $\lambda$ .

## II. TEVES THEORY IN THE SPACE-TIME GEOMETRY AROUND THE EARTH

The TeVeS theory coincides to the general relativity in two limits

- Milgrom limit:  $K \rightarrow 0$ ,  $l \rightarrow \infty$
- Bekenstein limit:  $k \rightarrow 0$ ,  $K \propto k$ ,  $l \propto k^{-\frac{3}{2}}$

The Bekenstein limit can be understood as the perturbative regime of the Milgrom limit. Here we consider the Bekenstein limit. We shall consider the Newtonian regime of TeVeS theory wherein  $\mu = 1$  and the contribution of  $F$  to the equation of motion of the metric can be neglected [3]. Note that the term neglected does not contribute to  $\frac{1}{\tau^2}$  behavior of the fields around spherical mass distribution we are interested in. So we have

$$\tau_{ab} \equiv \sigma^2 \left( \phi_{,a} \phi_{,b} - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} g_{ab} - A^\mu \phi_{,\mu} (A_{(a} \phi_{,b)} - \frac{1}{2} A^\nu \phi_{,\nu} g_{ab}) \right) \quad (13)$$

$$\sigma^2 = \frac{1}{kG} \quad (14)$$

We are interested in the solution representing the space-time geometry around the earth. The linear equations of motion suffice to describe the dynamics of gravity near the earth with the precision required. So we ignore all the non-linear terms in the field equations. To this aim we consider that

$$g_{ab} = \eta_{ab} + h_{ab} \quad (15)$$

and we keep the linear terms in  $h_{ab}$ . We also utilize the fact that  $h_{ab} \propto T_{ab}$ ,  $\phi \ll 1$  and keep the linear leading terms:

$$G_{ab} = 8\pi GT_{ab} - \lambda A_a A_b \quad (16)$$

$$K \nabla^b F_{ab} + \lambda A_a = 0 \quad (17)$$

$$\nabla_b ((g^{ab} - A^a A^b) \nabla_a \phi) = kG[g^{ab} + 2A^a A^b]T_{ab} \quad (18)$$

Contracting (17) with  $A^a$  and recalling  $A_a A^a = -1$ , results

$$\lambda = K A^a \nabla^b F_{ab} \quad (19)$$

inserting which back into (17) gives a second order non-linear equation for  $A$ :

$$\nabla^\mu F_{a\mu} + A^\mu \nabla^\nu F_{\mu\nu} A_a = 0 \quad (20)$$

We require spatial isotropy far away from the matter distribution. The isotropy demands

$$\bar{A}_a = (1, 0, 0, 0) \quad (21)$$

$$A_{(0)}^a = \bar{A}_a \quad (22)$$

The presence of the matter adds a correction to this value of  $A_{(0)}^a$ :

$$A_{(0)}^a = \bar{A}_a + \delta A^a \quad (23)$$

$$\delta A^a = \sum_{n=1}^{\infty} \delta_n A^a \quad (24)$$

where  $\delta_n A^a$  represents the term at the order of  $(h_{ab})^n$ . We are interested only in the linear correction to  $A$ :  $\delta_1 A$ . Since the field strength of the  $\bar{A}$  vanishes,  $\delta_1 A^a$  does not couple to  $h_{ab}$ . Therefore  $\delta_1 A^a = 0$  which in turn implies that  $\lambda$  is vanishing at the linear order approximation to the equations of motion:

$$\lambda = 0 + O(h^2) \quad (25)$$

Due to the value of  $\lambda$  (25), the equation of motion for the metric (16) simplifies to the ordinary Einstein equation:

$$G_{ab} = 8\pi GT_{ab} \quad (26)$$

We are interested in the gravitomagnetism approximation to general relativity. So we consider the physical energy momentum of a pressure-less slow moving mass distribution where the non-zero components of the energy momentum tensor read

$$T_{00} = \rho \quad (27)$$

$$T_{0i} = J_i = \rho v_i \quad (28)$$

where  $\rho$  is physical density and  $v_i$  represents the physical velocity. Defining the Einstein gravitoelectric and gravitomagnetic potential as usual

$$g_{00} = 1 + 2\Phi \quad (29a)$$

$$g_{0i} = \mathcal{A}_i \quad (29b)$$

These quantities are solved to

$$\Phi = \Phi_N \quad (30a)$$

$$\mathcal{A}_i = A_i^{EH} \quad (30b)$$

where  $\Phi_N$  is the Newtonian field potential and  $A_i^{EH}$  is the gravitomagnetic potential of the Einstein-Hilbert theory.

In the absence of the mass distribution, we require the physical metric to coincide to the Einstein metric. This sets the cosmological value of the scalar field to zero. This choice corresponds to the cosmological definition of time where in the physical time and the Einstein time coincides to each other. In the presence of matter  $\phi$  also gets a series expansion in term of the deviation from the Minkowski metric

$$\delta\phi = \sum_{n=1}^{\infty} \delta_n \phi \quad (31)$$

where  $\delta_n \phi$  represents the term at the order of  $(h_{ab})^n$ . We assume that  $v_i \ll 1$  so the equation of motion of  $\delta_1 \phi$  can be approximated to

$$\nabla_i \nabla^i \delta_1 \phi = kG \rho \quad (32)$$

This implies that

$$\delta_1 \phi = \frac{k}{4\pi} \Phi_N \quad (33)$$

where  $\Phi_N$  as in the previous case represent the Newtonian gravitational potential. Due to the way that the scalar appears in the physical metric, beside the fact that  $\bar{A}$  has only a temporal component, the contribution of  $J$  to  $\delta_1 \phi$  - which we have neglected - does not contribute to the  $\tilde{g}_{0i}$  components of the physical metric.

The physical gravitomagnetic and gravitoelectric fields are defined by the metric that a test particle probes, or the metric that appears in the worldline action of a particle

$$S = \int d\tau \tilde{g}_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \quad (34)$$

where  $\tau$  is an affain parameter. It is the physical metric that leads to the physical gravitomagnetic and gravitoelectric force. Inserting (33), (29) and (30) into the definition of the physical metric (1), the non-vanishing components of the physical metric at the gravitomagnetism approximation follows

$$\tilde{g}_{00} = 1 - 2\left(1 + \frac{3k}{4\pi}\right)\Phi^N \quad (35)$$

$$\tilde{g}_{0i} = \mathcal{A}_i \quad (36)$$

Therefore the gravitomagnetic and gravitoelectric field of the TeVeS theory read

$$\Phi_{TeVes} = \left(1 + \frac{3k}{4\pi}\right)\Phi^N \quad (37)$$

$$\mathcal{A}^{TeVes} = \mathcal{A}_i^{EH} \quad (38)$$

The value of the Newtonian constant is identified by fitting  $1/r^2$  function on the gravitoelectric acceleration around stationary mass distribution. So the reported value of  $G$  is in fact  $G_{obs} = G(1 + \frac{3k}{4\pi})$ . Let the gravitomagnetic/electric field be expressed in term of the observed value of the gravitational Newton constant:

$$\Phi_{TeVes} = \Phi_{obs}^N \quad (39)$$

$$\mathcal{A}^{TeVes} = \left(1 - \frac{3k}{4\pi}\right)\mathcal{A}_i^{EH} \quad (40)$$

This means that the gravitomagnetic field of the TeVeS solution representing the space-time geometry around the earth is weaker than the Einstein-Hilbert theory's prediction by the factor of  $\frac{3k}{4\pi}$ .

Ref. [6] considers the lunar laser ranging measurements and reports that the Einstein-Hilbert gravitomagnetic force is consistent with the observation with 0.1% accuracy. This bound requires

$$\frac{k}{4\pi} < 3.3 \times 10^{-4} \quad (41)$$

at the standard confidence level. This bound is two orders of magnitude weaker than what is commonly expected for  $k$  in order to keep the TeVeS theory in its MOND regime consistent with the cosmological/astronomical observation. However, a priori contradicting this expectation does not diminish the possibility to retain the theory consistent with the nature. The Gravity Probe B [7] through the Pugh-Schiff effect [8, 9] provides independent measurements of the geodetic and frame-dragging effects at an accuracy of 0.28% and 19%, respectively, neither of which improves the LLR measurements' constraint on  $k$ .

Now that we have a direct constraint on  $k$ , we can constraint the parameter of  $l$  in TeVeS theory. Noting that  $K < 10^{-2}$ , the MOND limit of the TeVeS theory for the choice of the  $F$  function of [3], gives to the following critical acceleration

$$a_0 \equiv \frac{\sqrt{3kc^2}}{4\pi l} \quad (42)$$

The MOND terminology requires [10]

$$a_0 = (1.0 \pm 0.2) \times 10^{-10} \frac{m}{s^2}. \quad (43)$$

Our constraint on  $k$  then demands that

$$l > (1.65 \pm 0.35) \times 10^{25} m \quad (44)$$

at the standard confidence level. This is the first constraint on the dimension full parameter of the TeVeS theory.

- 
- [1] M. Milgrom, *Astrophysical Journal* **270** (1983) 365 ; *Astrophysical Journal* **270** (1983)371
  - [2] J. Bekenstein, M. Milgrom, *Astrophysical J.* **286**, (1984) 7.
  - [3] J. D. Bekenstein, *Phys. Rev. D* **70** (2004) 083509 [Erratum-ibid. *D* **71** (2005) 069901] [arXiv:astro-ph/0403694].
  - [4] Q. Exirifard, arXiv:1107.2109 [gr-qc].
  - [5] D. Giannios, *Phys. Rev. D* **71** (2005) 103511 [gr-qc/0502122].
  - [6] T. W. Murphy, Jr., K. Nordtvedt, S. G. Turyshev, *Phys. Rev. Lett.* **98** (2007) 071102. [gr-qc/0702028].
  - [7] C. W. F. Everitt *et al.*, *Phys. Rev. Lett.* **106** (2011) 221101, arXiv:1105.3456 [gr-qc].
  - [8] Pugh, G.E.: WSEG Research Memorandum No. **11** (1959).
  - [9] L. I. Schiff, *Phys. Rev. Lett.* **4**, 215 (1960).
  - [10] J. D. Bekenstein, arXiv:astro-ph/0701848; K. G. Begeman, A. H. Broeils and R. H. Sanders, *Mon. Not. Roy. Astron. Soc.* **249** (1991) 523.